

# FUSION ALGEBRAS WITH NEGATIVE STRUCTURE CONSTANTS

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**ABSTRACT.** We introduce fusion algebras with not necessarily positive structure constants and without identity element. We prove that they are semisimple when tensored with  $\mathbb{C}$  and that their characters satisfy orthogonality relations. Then we define the proper notion of subrings and factor rings for such algebras. For certain algebras  $R$  we prove the existence of a ring  $R'$  with nonnegative structure constants such that  $R$  is a factor ring of  $R'$ . We give some examples of interesting factor rings of the representation ring of the quantum double of a finite group. Then, we investigate the algebras associated to Hadamard matrices. For an  $n \times n$ -matrix the corresponding algebra is a factor ring of a subalgebra of  $\mathbb{Z}[(\mathbb{Z}/2\mathbb{Z})^{n-2}]$ .

## 1. INTRODUCTION

A  $\mathbb{Z}$ -based ring is a finite dimensional  $\mathbb{Z}$ -algebra with certain properties which make it semisimple when tensored with  $\mathbb{C}$ . Actually, it is almost a fusion algebra (or table algebra [1], or based ring [12]) in the most common sense except that it may have negative structure constants (and is commutative). The fact that the structure constants are all nonnegative in fusion algebras has important consequences which we miss in the case of  $\mathbb{Z}$ -based rings. For example, the corresponding  $s$ -matrix (an analogue of a character table) will always have a column with only nonzero positive real entries which are sometimes called degrees. This is not the case for example for the algebras belonging to the imprimitive “spetses” (see [8], [14] and [5] or [15] for a definition of spetses).

There are other reasons why algebras with negative structure constants are worth to be investigated. Some of the Fourier matrices corresponding to the exceptional “spetses” may be constructed using factor rings of some fusion algebras and in general, “factor rings” (see 3.2 for an exact definition in the case of  $\mathbb{Z}$ -based rings) will often have negative structure constants.

For a given  $\mathbb{Z}$ -based ring  $R$ , it is an interesting task to find a fusion algebra  $R'$  such that  $R$  is a factor ring  $R'/I$  for some ideal  $I$ . In some cases, it suffices to double the basis (add “negatives”), but in general, it is not clear if this is possible. We prove the existence of such a ring  $R'$  for certain  $\mathbb{Z}$ -based rings:

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*Key words and phrases.* fusion algebra, table algebra, Hadamard matrix.

**Theorem 1.1.** *Let  $R$  be a  $\mathbb{Z}$ -based rng with  $s$ -matrix  $s$ . If  $s$  is of the form*

$$s_{ij} = \mu_j v_{ij}$$

*where  $0 \neq \mu_j \in \mathbb{Z}$  and  $v_{ij}$  is a root of unity or 0, then  $R$  is a factor ring of a pointed  $\mathbb{Z}$ -algebra with nonnegative structure constants (an algebra with a distinguished basis).*

The proof is given in section 3. In particular, it follows that the ring associated to a Hadamard matrix is a factor ring of a ring with nonnegative structure constants.

Motivated by exterior powers of character rings and by Hadamard matrices, we generalize the definition of a  $\mathbb{Z}$ -based ring by dropping the assumption that the algebra should have a one. It appears that most properties of table algebras can be transferred to such “ $\mathbb{Z}$ -based rngs” (the missing “i” is meant to suggest that it is a “ring” without an identity element).

The concept of closed subsets that has been introduced for table algebras (see for example [4], [3], [1]) works in the case of  $\mathbb{Z}$ -based rings too ([7]). And it may also be generalized to  $\mathbb{Z}$ -based rngs although it is not trivial that the span of a subset of the basis which is closed under multiplication is again a  $\mathbb{Z}$ -based rng. We prove (see section 2.6):

**Theorem 1.2.** *If the  $\mathbb{Z}$ -module spanned by a subset of the basis of a  $\mathbb{Z}$ -based rng is closed under multiplication, then it is a  $\mathbb{Z}$ -based rng.*

An  $n \times n$  Hadamard matrix can be interpreted as an  $s$ -matrix (after an adequate rescaling), it yields a  $\mathbb{Z}$ -algebra which is a  $\mathbb{Z}$ -based rng. If  $n = 4k$  with  $k > 1$  odd, then this algebra will always have some negative structure constants. It is a factor ring of a subalgebra of the group algebra  $\mathbb{Z}[Z_2^{n-2}]$ , where  $Z_2$  denotes the cyclic group with two elements.

In the case of Hadamard matrices, the structure constants are a subset of the numbers used to define profiles (see [17], 9.3). Profiles are useful to test if Hadamard matrices are equivalent. The structure constants contain enough information to reconstruct the matrix. This is a consequence of the theorem of Wedderburn-Artin in the case of arbitrary  $\mathbb{Z}$ -based rngs, and for Hadamard matrices this reduces to the fact that diagonalizable commuting matrices may be diagonalized simultaneously. Of course the resulting matrix is unique only up to permutations of rows and columns, which does not change the equivalence class. Multiplying rows or columns by signs is a more complicated operation and it is less obvious when two algebras are equivalent via sign changes, although there are good algorithms to find this out.

The structure of the paper is as follows:

In the first section we introduce the  $\mathbb{Z}$ -based rngs and its  $s$ -matrix and prove orthogonality relations. Then we define closed subsets for  $\mathbb{Z}$ -based rngs and prove that for any basis element  $b$ , there exists a power  $b^m$  which has non zero value under the symmetrizing trace  $\tau$ . This enables us to prove

that closed subsets span  $\mathbb{Z}$ -based rngs. We also briefly describe an algorithm for finding closed subsets.

Then we study ideals and factor rings, the objective being to realize  $\mathbb{Z}$ -based rngs as factor rings of fusion algebras. We give an example of such ideals for the case of the representation ring of the quantum double of a finite group and cite some examples of factor rings of algebras coming from Kac-Moody algebras in the following section. In the last section we apply our theory to the rings associated to Hadamard matrices.

## 2. $\mathbb{Z}$ -BASED RINGS WITHOUT ONE

In [8] the so called  $\mathbb{Z}$ -based ring has been introduced. This is a commutative  $\mathbb{Z}$ -algebra  $R$  given with a basis  $B$  and an involution  $\sim$  with certain properties which enable us to show that  $R_{\mathbb{C}} := R \otimes_{\mathbb{Z}} \mathbb{C}$  is semisimple. One important axiom in the definition is that the unit of the algebra is an element of  $B$ . We would like to preserve the property that  $R_{\mathbb{C}}$  is semisimple without requiring  $1 \in B$ .

In the following, we embed  $R$  into  $R_{\mathbb{C}}$  by  $r \mapsto r \otimes 1$  and denote  $r \otimes 1$  by  $r$  (this is possible because  $R$  is a free  $\mathbb{Z}$ -module).

**2.1. Definition.** Let  $R$  be a finitely generated commutative  $\mathbb{Z}$ -algebra which is a free  $\mathbb{Z}$ -module with basis  $B = \{b_0, \dots, b_{n-1}\}$  and *structure constants*

$$b_i b_j = \sum_m N_{ij}^m b_m, \quad N_{ij}^m \in \mathbb{Z}$$

for  $0 \leq i, j < n$ .

Assume that there is an  $e \in R_{\mathbb{C}}$  such that  $ea = a$  for all  $a \in R_{\mathbb{C}}$ . In this case,  $e$  is uniquely determined by this condition.

Let  $\tau_i : R \rightarrow \mathbb{C}$  be defined to be the linear extension of  $\tau_i(b_j) = \delta_{i,j}$ . If  $e = \sum_i e_i b_i$  in  $R_{\mathbb{C}}$  with  $e_i \in \mathbb{C}$ , then define

$$\tau := \sum_i \bar{e}_i \tau_i,$$

where  $\bar{\phantom{x}}$  means complex conjugation. Now assume the existence of an involution  $\sim : R \rightarrow R$  which is a  $\mathbb{Z}$ -module homomorphism such that

$$\tilde{B} = B, \quad N_{ij}^m = N_{\tilde{i}\tilde{j}}^{\tilde{m}}, \quad \tau(\tilde{b}_i b_j) = \delta_{i,j}$$

for all  $0 \leq i, j, m < n$ , where  $\tilde{i}$  is the index with  $\tilde{b}_i = b_{\tilde{i}}$ .

We extend  $\sim$  and  $\tau$  to  $R_{\mathbb{C}}$  by

$$\widetilde{r \otimes z} := \tilde{r} \otimes \bar{z}, \quad \tau(r \otimes z) := z\tau(r)$$

for  $r \in R$ ,  $z \in \mathbb{C}$  and use the same symbols for the extended maps. We require the following additional property of  $\sim$ :

$$\tilde{\tilde{e}} = e.$$

**Definition 2.1.** We call  $(R, B)$  with  $e$  and  $\sim$  as above a  *$\mathbb{Z}$ -based rng*.

The definition of  $\tau$  is motivated by  $\tau(\tilde{b}_i b_j) = \delta_{i,j}$ , because then

$$\tau(e) = \tau(\tilde{e}e) = \tau\left(\sum_{i,j} \bar{e}_i e_j \tilde{b}_i b_j\right) = \sum_i \bar{e}_i e_i$$

and

$$\tau(e) = \sum_i \bar{e}_i \tau_i(e) = \sum_i \bar{e}_i e_i.$$

Using  $e = \tilde{e}$  we also get  $\bar{e}_i = e_{\tilde{i}}$  and hence

$$\tau(\tilde{r}) = \sum_j \bar{e}_j \tau_j\left(\sum_i \bar{\mu}_i \tilde{b}_i\right) = \sum_j \bar{e}_j \bar{\mu}_{\tilde{j}} = \sum_j e_j \bar{\mu}_j = \overline{\sum_j \bar{e}_j \tau_j\left(\sum_i \mu_i b_i\right)} = \overline{\tau(r)}$$

for any  $r = \sum_i \mu_i b_i \in R_{\mathbb{C}}$ .

The map

$$\langle \cdot, \cdot \rangle : R \times R \rightarrow \mathbb{Z}, \quad \langle r, r' \rangle := \tau(\tilde{r}r')$$

for  $r, r' \in R$  behaves like an inner product with orthonormal basis  $B$  because  $r = \sum_{b \in B} \langle b, r \rangle b$  for all  $r \in R$ . Then  $\tilde{B}$  is the basis dual to  $B$  with respect to this inner product.

It is easily seen that extending  $\langle \cdot, \cdot \rangle$  to  $R_{\mathbb{C}} \times R_{\mathbb{C}}$  yields a hermitian positive definite sesquilinear form, so it is non degenerate.

**Proposition 2.2.** *Let  $R$  be a  $\mathbb{Z}$ -based rng. Then the algebra  $R_{\mathbb{C}} = R \otimes_{\mathbb{Z}} \mathbb{C}$  is semisimple.*

*Proof.* If  $\mathfrak{J}$  is an ideal in  $R_{\mathbb{C}}$ , then the orthogonal complement

$$\mathfrak{J}^{\perp} := \{r \in R_{\mathbb{C}} \mid \langle r, r' \rangle = 0 \quad \forall r' \in \mathfrak{J}\}$$

is a left ideal too:

$$\langle tr, r' \rangle = \tau(\tilde{t}rr') = \tau(\tilde{r}\tilde{t}r') = \langle r, \tilde{t}r' \rangle = 0$$

for all  $r \in \mathfrak{J}^{\perp}$ ,  $t \in R_{\mathbb{C}}$  and  $r' \in \mathfrak{J}$ . The claim follows.  $\square$

**2.2.  $s$ -matrix.** Now  $R_{\mathbb{C}}$  is a commutative semisimple algebra over an algebraically closed field, so by the theorem of Wedderburn-Artin it is isomorphic as a  $\mathbb{C}$ -algebra to  $\mathbb{C}^n$  with componentwise multiplication. By choosing  $B$  as a basis for  $R_{\mathbb{C}}$  and the canonical basis  $\{v_i\}_i$  with  $v_i v_j = \delta_{i,j} v_i$  for all  $i, j$  for  $\mathbb{C}^n$ , an isomorphism  $\varphi$  is described by a matrix  $s$  which we will call an  $s$ -matrix of  $(R, B)$ :

$$\varphi(b_i) = \sum_k s_{ki} v_k.$$

Remark that this matrix depends on the choice of the isomorphism  $\varphi$ . Another isomorphism would differ from  $\varphi$  by a  $\mathbb{C}$ -algebra automorphism of  $\mathbb{C}^n$ , so an  $s$ -matrix is unique up to a permutation of rows.

The algebra  $R$  is the  $\mathbb{Z}$ -lattice spanned by the columns of  $s$ . Its multiplication is componentwise multiplication of vectors and the involution  $\sim$

corresponds to complex conjugation on the columns (see lemma 2.6). The structure constants are (Verlinde's formula)

$$(1) \quad N_{ij}^m = \sum_l s_{li} s_{lj} s'_{ml}$$

where  $s' = s^{-1}$  (this is just the linear decomposition of the product of two columns with respect to the columns).

**2.3. Orthogonality relations.** The rows of  $s$  are the images of  $B$  under the one-dimensional representations of  $R$  because  $s_{ki} s_{kj} = \sum_l N_{ij}^l s_{kl}$  for all  $k, i, j$ . They are orthogonal. To prove this, we need:

**Lemma 2.3.** *Let  $E_1, E_2$  be  $R_{\mathbb{C}}$ -modules and  $h : E_1 \rightarrow E_2$  be a  $\mathbb{C}$ -linear map. Then the map  $h_0 : E_1 \rightarrow E_2, w \mapsto \sum_{i \in I} b_i h(\tilde{b}_i w)$  is  $R_{\mathbb{C}}$ -linear.*

*Proof.* For  $j, k, m \in \{0, \dots, n-1\}$ , we have

$$\tau(\tilde{b}_m \sum_i N_{jk}^i b_i) = \sum_i N_{jk}^i \tau(\tilde{b}_m b_i) = N_{jk}^m,$$

so  $\tau(b_m b_j b_k) = N_{jk}^m$ . Because of  $\tau(\tilde{r}) = \overline{\tau(r)}$ , we get

$$N_{jk}^i = \tau(\tilde{b}_i \tilde{b}_j b_k) = \overline{\tau(\tilde{b}_k b_j \tilde{b}_i)} = \overline{N_{ji}^k} = N_{ji}^k.$$

Now we can conclude that  $h_0$  is a homomorphism of  $R_{\mathbb{C}}$ -modules: for  $r = \sum_k \mu_k \tilde{b}_k \in R_{\mathbb{C}}$  and  $w \in E_1$ , we have

$$\begin{aligned} h_0(r \cdot w) &= \sum_i b_i h(\tilde{b}_i \sum_k \mu_k \tilde{b}_k w) \stackrel{(*)}{=} \sum_i b_i h(\sum_k \mu_k \sum_j N_{ki}^j \tilde{b}_j w) \\ &= \sum_k \mu_k \sum_{j,i} N_{ki}^j b_i h(\tilde{b}_j w) = \sum_k \mu_k \sum_{j,i} N_{kj}^i b_i h(\tilde{b}_j w) \\ &= \sum_k \mu_k \tilde{b}_k \sum_j b_j h(\tilde{b}_j w) = r \cdot h_0(w), \end{aligned}$$

where  $(*)$  holds because  $\tilde{b}_i \tilde{b}_k = \widetilde{\tilde{b}_k b_i} = \sum_j \widetilde{N_{ki}^j b_j} = \sum_j N_{ki}^j \tilde{b}_j$ .  $\square$

This implies:

**Proposition 2.4.** *Let  $R$  be a  $\mathbb{Z}$ -based rng. Then its  $s$ -matrix has orthogonal rows.*

*Proof.* Let  $E_1, E_2$  be one dimensional  $R_{\mathbb{C}}$ -modules and  $\chi_1, \chi_2$  be the corresponding characters (trace of the operation of  $R$  on  $E_1, E_2$ ). So  $b \in B$  acts on  $E_1$  respectively  $E_2$  as the scalar  $\chi_1(b)$  respectively  $\chi_2(b)$ . For an arbitrary  $\mathbb{C}$ -linear map  $h : E_2 \rightarrow E_1$ , lemma 2.3 states that the map  $h_0$  given by

$$h_0(w) = \sum_{b \in B} \chi_1(b) h(\chi_2(\tilde{b}) w) = \left( \sum_{b \in B} \chi_1(b) \chi_2(\tilde{b}) \right) h(w)$$

is a homomorphism of  $R_{\mathbb{C}}$ -modules from  $E_2$  to  $E_1$ . But these modules are irreducible, so by the lemma of Schur,  $h_0$  is either an isomorphism or the zero map. This holds for all  $\mathbb{C}$ -linear maps  $h$ , hence

$$\sum_{b \in B} \chi_1(b) \chi_2(\tilde{b})$$

is zero if  $E_1 \not\cong E_2$ , i.e.  $\chi_1 \neq \chi_2$ . On the other hand, if  $h_0$  is an isomorphism, then the above sum can not be zero, because there is at least one element in  $E_2$  which is not mapped to zero.  $\square$

**Definition 2.5.** By normalizing the rows of the  $s$ -matrix we get an orthonormal matrix called the *Fourier matrix* of  $(R, B)$ .

The Fourier matrix is unique if we choose the positive square roots of the rows in the normalization.

#### 2.4. Examples.

*Example 1.* Of course, a  $\mathbb{Z}$ -based ring is also a  $\mathbb{Z}$ -based rng. So for example representation rings of certain Hopf algebras or exterior powers of group rings of cyclic groups are  $\mathbb{Z}$ -based rngs (see [8] or [7], 5.1).

*Example 2.* Let  $s = kI$  be a scalar matrix,  $k \in \mathbb{Z}$ . Then  $b_i b_j = k \delta_{ij} b_i$ . In  $R_{\mathbb{C}}$ ,  $e = \sum_i \frac{1}{k} b_i$ . The involution  $\sim$  is the identity map. Further,  $\tau = \sum_i \frac{1}{k} \tau_i$  and indeed,  $\tau(b_i b_i) = 1$  for all  $i$ .

*Example 3.* Let  $s$  be of the form

$$s = \begin{pmatrix} k & & \\ \vdots & * & \\ k & & \end{pmatrix}.$$

Then  $e = \frac{1}{k} b_0$ ,  $\tau = \frac{1}{k} \tau_0$ . Suppose  $\sim$  is the identity map. Then  $\tau(b_i b_i) = 1$  for all  $i$ . So  $\tau_0(b_i b_i) = k$  for all  $i$ .

*Example 4.* The matrix

$$s := \begin{pmatrix} -2 & 0 & 2 & -2 & 0 & -2 \\ 0 & -2 & -2 & -2 & -2 & 0 \\ 2 & -2 & 0 & 0 & 2 & -2 \\ -2 & -2 & 0 & 0 & 2 & 2 \\ 0 & -2 & 2 & 2 & -2 & 0 \\ -2 & 0 & -2 & 2 & 0 & -2 \end{pmatrix}$$

is the exterior square of the character table of  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ . By the formula of Verlinde, it defines structure constants for a  $\mathbb{Z}$ -based rng. We have  $\tau = -\frac{1}{4}(\tau_0 + 2\tau_1 + \tau_5)$ .

*Example 5.* Consider the matrix

$$s := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The lattice spanned by the columns of  $s$  is closed under componentwise multiplication but it is not a  $\mathbb{Z}$ -based rng because there is no adequate

involution  $\sim$ . Remark that  $s$  is not orthogonal. This is the “ $s$ -matrix” of the monoid ring  $\mathbb{Z}[(\mathbb{Z}/2\mathbb{Z})^2]$  where  $(\mathbb{Z}/2\mathbb{Z})^2$  is a monoid by componentwise multiplication.

### 2.5. Complex conjugation on the columns of $s$ .

**Lemma 2.6.** *Let  $s$  be the  $s$ -matrix of a  $\mathbb{Z}$ -based rng  $R$ . Then the set of columns  $\{s_0, \dots, s_{n-1}\}$  of  $s$  is closed under complex conjugation and  $s_{\bar{i}} = \bar{s}_i$  for all  $i$ .*

*Proof.* The matrix  $s$  is orthogonal, so  $D := s\bar{s}^T$  is diagonal. Let  $d$  be the vector with entries  $d_i := D_{ii}^{-1}$  and denote by  $\bar{b}$  the complex conjugate of a vector  $b \in \mathbb{C}^n \cong R_{\mathbb{C}}$ . Then we have (remember that  $e = \sum_i e_i b_i$ ,  $\tau = \sum_i \bar{e}_i \tau_i$ )

$$e_i = \sum_j \bar{s}_{ji} d_j$$

and hence

$$\tau(b_i \bar{b}_u) = \sum_j \bar{e}_j \tau_j(b_i \bar{b}_u) = \sum_j \bar{e}_j \sum_l s_{li} \bar{s}_{lu} \bar{s}_{lj} d_j = \sum_{j,m,l} s_{mj} d_m s_{li} \bar{s}_{lu} \bar{s}_{lj} d_j.$$

Since the rows  $l$  and  $m$  of  $s$  are orthogonal by proposition 2.4,

$$\tau(b_i \bar{b}_u) = \sum_m d_m s_{mi} \bar{s}_{mu} = \delta_{iu}$$

which implies  $\tilde{b}_i = \bar{b}_i$  for all  $i$ .  $\square$

Let  $\zeta$  be a primitive  $q$ -th root of unity,  $q \in \mathbb{N}$ . We will need the following proposition later on.

**Proposition 2.7.** *Let  $(R, B)$  be a  $\mathbb{Z}$ -based rng with  $s$ -matrix  $s$  with columns of the form  $s_i = \mu_i v_i$ ,  $v_i \in C'^n$ ,  $C' = \{\zeta, \dots, \zeta^q\}$ ,  $\mu_i \in \mathbb{Z}$ . Then  $|\mu_i| = |\mu_j|$  for all  $i, j$ .*

*Proof.* Since  $v_i^{-1} = \bar{v}_i = \frac{1}{\mu_i} s_{\bar{i}}$  by lemma 2.6, for  $\eta := \tau(e) = \tau(v_i v_i^{-1})$  we get

$$1 = \tau(b_i \tilde{b}_i) = \mu_i^2 \tau(v_i v_i^{-1}) = \mu_i^2 \eta$$

i.e.  $\mu_i = \pm \sqrt{\eta^{-1}}$  for all  $i$ .  $\square$

### 2.6. Closed subsets.

**Definition 2.8.** Let  $(R, B)$  be a  $\mathbb{Z}$ -based rng. A subset  $B' \subset B$  is called *closed*, if the  $\mathbb{Z}$ -module spanned by  $B'$  is closed under the multiplication of  $R$ .

*Example 6.* The closed subsets of a character ring of a finite group  $G$  are in bijection with the normal subgroups of  $G$ . The subalgebra corresponding to  $N \trianglelefteq G$  is isomorphic to the character ring of the factor group  $G/N$ . This is a corollary to a theorem of Burnside and Brauer (see [7], 3.1.2).

We prove that the subalgebra given by a closed subset is a  $\mathbb{Z}$ -based rng. First, we need:

**Lemma 2.9.** *If  $(R, B)$  is a  $\mathbb{Z}$ -based rng and  $b \in B$ , then there exists an  $m \in \mathbb{N}$  such that  $\tau(b^m) \neq 0$ .*

*Proof.* Let  $v$  be the column of the  $s$ -matrix  $s$  of  $R$  corresponding to  $b$ . If  $v^m = \sum_i \mu_i x_i$ , where  $x_i$  are the columns of  $s$ , then  $\tau(b^m) = \sum_i \bar{e}_i \mu_i$ . Let  $w$  be the vector with the inverses of the norms of the rows of  $s$  as entries, so it has positive non zero real numbers as entries.

The  $X_i := wx_i$  (componentwise multiplication) are orthonormal, because the rows of  $s$  are orthogonal. The sum  $e = \sum_i e_i x_i$  is the linear combination that gives the vector with all entries 1 (because it is the unit of the algebra).

Now  $wv^m = \sum_i \mu_i wx_i = \sum_i \mu_i X_i$ , and because the  $X_i$  are orthonormal, we get

$$\tau(b^m) = \sum_i e_i \mu_i = \left( \sum_i \mu_i X_i \mid \sum_j e_j X_j \right) = (wv^m \mid w),$$

( $w = \bar{w}$ ,  $e = \bar{e}$ ,  $(\cdot \mid \cdot)$  is the standard inner product on  $\mathbb{C}^{|B|}$ ).

If the basis has  $n$  elements, then the powers  $v, v^2, \dots, v^{n+1}$  of  $v$  are linearly dependent. Let  $v'$  be the vector

$$v'_i := \begin{cases} v_i^{-1} & \text{if } v_i \neq 0 \\ 0 & \text{if } v_i = 0 \end{cases}$$

and  $\pi$  be the projection onto the subspace of  $\mathbb{C}^{|B|}$  given by the nonzero entries of  $v$ , so  $\pi : u \mapsto vv'u$ . Choose  $k$  minimal such that  $\pi(e) = vv', v^1, v^2, \dots, v^k$  are linearly dependent. Then there exist  $c_0, \dots, c_{k-1} \in \mathbb{C}$  with

$$v^k + c_{k-1}v^{k-1} + \dots + c_1v + c_0\pi(e) = 0.$$

Now suppose  $\tau(b^m) = 0$  for all  $m > 0$ . Then  $(wv^m \mid w) = 0$  for all  $m > 0$  and in particular

$$\begin{aligned} 0 &= \langle w(v^k + c_{k-1}v^{k-1} + \dots + c_1v + c_0\pi(e)), w \rangle \\ &= \langle wc_0\pi(e), w \rangle \\ &= c_0 \langle \pi(w), \pi(w) \rangle, \end{aligned}$$

i.e.  $c_0 = 0$ . Then

$$v'(v^k + c_{k-1}v^{k-1} + \dots + c_1v) = v^{k-1} + c_{k-1}v^{k-2} + \dots + c_1\pi(e) = 0$$

in contradiction to the minimal choice of  $k$ .  $\square$

The last proof also shows that the entries of the  $s$ -matrix are algebraic over  $\mathbb{Q}$ , because the powers of a column are linearly dependent. We have more:

**Proposition 2.10.** *The entries of the  $s$ -matrix of a  $\mathbb{Z}$ -based rng are algebraic integers over  $\mathbb{Z}$ .*

*Proof.* If  $c_0, \dots, c_{n-1}$  are the entries of a row of  $s$ , then  $\mathbb{Z}[c_0, \dots, c_{n-1}]$  is finitely generated as a  $\mathbb{Z}$ -module, since  $R$  is isomorphic to the lattice spanned

by the columns of  $s$  with componentwise multiplication, and it is a free  $\mathbb{Z}$ -module of rank  $n$ . By Theorem [16], I.2.2, this is equivalent to the fact that  $c_0, \dots, c_{n-1}$  are integral over  $\mathbb{Z}$ .  $\square$

Presumably there exists an  $m$  with  $\tau(b^m) \in \mathbb{R}_{>0}$ . But it does not suffice to choose an  $m$  such that  $\tau(b^m) \in \mathbb{R}_{<0}$  and to take the square of  $b^m$ .

*Example 7.* If  $R := \mathbb{Z}[\mathbb{Z}/3\mathbb{Z}]$  with the group elements  $b_0 = 1, b_1, b_2$  as a basis, then  $\tilde{b}_1 = b_2$  and  $b_1 b_2 = 1$ . We have  $\tau(-b_0 - b_1 + b_2) = -1$  and  $\tau((-b_0 - b_1 + b_2)^2) = \tau(-b_0 + 3b_1 - b_2) = -1$ . However,  $\tau((-b_0 - b_1 + b_2)^3) = 5 > 0$ .

**Proposition 2.11.** *Let  $(R, B)$  be a  $\mathbb{Z}$ -based rng and  $R'$  the subalgebra generated by a closed subset  $B' \subseteq B$ . Then  $\widetilde{R'} \subseteq R'$ , i.e., closed subsets are  $\sim$ -invariant.*

*Proof.* It suffices to check that  $\tilde{b} \in B'$  for all  $b \in B'$ . For this, take  $b \in B'$  and  $m \in \mathbb{N}$  such that  $\tau(b^m) \neq 0$  (lemma 2.9). If  $b^{m-1} = \sum_{b' \in B'} c_{b'} b'$  for some  $c_{b'} \in \mathbb{Z}$ , then

$$0 \neq \tau(b^m) = \tau\left(\sum_{b'} b c_{b'} b'\right) = \sum_{b'} c_{b'} \tau(b b') = c_{\tilde{b}},$$

i.e.  $\tilde{b} \in B'$  and hence  $\widetilde{R'} \subseteq R'$ .  $\square$

The  $\mathbb{Z}$ -module spanned by the complement of a closed subset  $B'$  with subalgebra  $R'$  is an  $R'$ -module:

**Proposition 2.12.** *If  $(R, B)$  is a  $\mathbb{Z}$ -based rng and  $R'$  a subalgebra spanned by a closed subset  $B' \subseteq B$ , then the  $\mathbb{Z}$ -module spanned by  $B \setminus B'$  is an  $R'$ -module with respect to the multiplication of  $R$ .*

*Proof.* Let  $B'' := B \setminus B'$  and  $b \in B', m \in B''$ . If  $bm = ca + r$  with  $a \in B', c \in \mathbb{Z}$  and  $r \in \langle B \setminus \{a\} \rangle$  then

$$\tau(\tilde{a}bm) = \tau(\tilde{a}(ca + r)) = \tau(c\tilde{a}a) + \tau(\tilde{a}r) = c,$$

( $\tau(\tilde{a}r) = 0$ ). So  $\tilde{a}b = c\tilde{m} + r'$  for some suitable  $r'$  with  $\tau(\tilde{m}r') = 0$ . But since  $\tilde{a}, b$  are in  $B'$  and  $\tilde{m}$  is in  $B''$  (proposition 2.11),  $c$  has to be zero. This holds for all  $a \in B'$  and  $bm$  therefore lies in the span of  $B''$ .  $\square$

Now we can prove:

**Theorem 2.13.** *The subalgebra of a  $\mathbb{Z}$ -based rng spanned by a closed subset is a  $\mathbb{Z}$ -based rng.*

*Proof.* Let  $(R, B)$  be a  $\mathbb{Z}$ -based rng and  $B' \subseteq B$  be a closed subset generating a subalgebra  $R'$ . By proposition 2.11, we may restrict  $\tau$  and  $\sim$  to  $R'$ . It remains to check that we have an identity element in  $R'_\mathbb{C} = R' \otimes_\mathbb{Z} \mathbb{C}$ . In  $R_\mathbb{C}$  we have the element  $e = \sum_{b \in B} e_b b$ , and for  $r = \sum_{b' \in B'} \mu_{b'} b' \in R'$  we get

$$r = er = \sum_{b \in B} e_b \sum_{b' \in B'} \mu_{b'} b' b = \sum_{b, b'' \in B} \sum_{b' \in B'} e_b \mu_{b'} N_{b'b}^{b''} b''$$

where  $\sum_{b \in B} \sum_{b' \in B'} e_b \mu_{b'} N_{b'b}^{b''} = 0$  for  $b'' \notin B'$  because  $r \in R'$ . By proposition 2.12,  $N_{b'b}^{b''} = 0$  if  $b', b'' \in B'$  and  $b \notin B'$ . So

$$r = \sum_{b, b'', b' \in B'} e_b \mu_{b'} N_{b'b}^{b''} b'' = \sum_{b \in B'} e_b \sum_{b' \in B'} \mu_{b'} b' b = e' r$$

with  $e' := \sum_{b \in B'} e_b b$ . □

Closed subsets may be read of the  $s$ -matrix (in the same way as character rings of factor groups of a finite group may be found in its character table):

**Proposition 2.14.** *Let  $(R, B)$  be a  $\mathbb{Z}$ -based rng with  $s$ -matrix  $s$  and  $B' \subseteq B$  be a closed subset. Then if  $u_i$ ,  $i \in \{0, \dots, |B| - 1\}$  are the rows of the submatrix  $u$  of  $s$  consisting of the columns  $s_i$ ,  $i \in B'$ , then the matrix  $u'$  with rows  $\{u_i \mid 0 \leq i < |B|, u_i \neq 0\}$  (in any ordering) is an  $s$ -matrix of the subring  $R'$  spanned by  $B'$ .*

*Proof.* The rows of  $s$  are the irreducible representations of  $R$  evaluated on  $B$ . Restricting them to  $R'$  yields representations  $u_i$ ,  $i \in \{0, \dots, |B| - 1\}$  of  $R'$  (possibly  $u_i = u_j$  for  $i \neq j$ ,  $u_i = 0$  may also occur). Now if  $m := |\{u_i \mid 0 \leq i < |B|, u_i \neq 0\}|$  was smaller than  $|B'|$ , then the rank of  $u$  would be less than  $|B'|$ . But this is impossible because  $s$  is invertible. On the other hand,  $R'$  is a  $\mathbb{Z}$ -based rng by Theorem 2.13, so it has  $|B'|$  irreducible representations and not more, i.e.  $m = |B'|$  and  $u'$  is an  $s$ -matrix. □

A converse to the previous result also holds:

**Proposition 2.15.** *If  $(R, B)$  is a  $\mathbb{Z}$ -based rng with  $s$ -matrix  $s$  and  $B'$  is a subset of  $B$  such that the set of rows  $\neq 0$  of the submatrix  $u$  of  $s$  consisting of the columns of  $s$  indexed by  $B'$  has exactly  $|B'|$  elements, then  $B'$  is a closed subset.*

*Proof.* Let  $B' \subseteq B$  be a subset as in the assumptions and  $u_i$  be the rows of the submatrix  $u$  of  $s$ . We partition the set of indices of rows of  $u$  not equal to 0 in a disjoint union of sets  $T_1, \dots, T_r$ ,  $r \in \mathbb{N}$  such that  $u_i = u_j$  for  $i, j \in T_l$  and  $u_i \neq u_j$  for  $i \in T_l$ ,  $j \in T_m$  with  $l \neq m$  for all  $i, j, l, m$ . Let  $w_l$  be the vector with 1's at the entries  $i \in T_l$  and 0 else. Then

$$V := \langle w_1, \dots, w_r \rangle$$

is a vector space of dimension  $r$  and  $r = |B'|$  because there are  $|B'|$  different rows in  $u$ . The columns of  $u$  are obviously elements of  $V$ . But the product of two columns of  $u$  also lies in  $V$ , because all entries to a  $T_l$  remain equal. Further, the rank of  $u$  is  $r$  since  $s$  is invertible, so  $V$  is equal to the space spanned by the columns of  $u$ . So the linear decomposition  $\sum_l N_{ij}^l s_l$  of the product of two columns  $s_i, s_j$  of  $u$  with respect to the columns of  $s$  has only coefficients  $N_{ij}^l$  not equal to 0 for  $l \in B'$  which means that  $B'$  is a closed subset. □

The last two propositions suggest an algorithm for a heuristic search for closed subsets. Start with two rows of an  $s$ -matrix  $s$  and consider the set  $B'$  of indices for which these rows have the same entry. Then count the different rows in the submatrix consisting of the columns of  $s$  indexed by  $B'$ . If this number is  $|B'|$  then  $B'$  is a closed subset.

We do not find all closed subsets by this method in general. But we can improve the algorithm by adding the intersection of any two such sets and iterate this procedure until there are no further intersections left. But there may still exist some closed sets that we do not find with this algorithm (remark that these subsets are rare in our applications).

*Example 8.* The exterior square  $s$  of the  $s$ -matrix of the group ring  $\mathbb{Z}[(\mathbb{Z}/2\mathbb{Z})^3]$  is an  $s$ -matrix of a  $\mathbb{Z}$ -based rng of rank 28. It has a closed subset with 16 elements which may not be found by our algorithm.

### 3. IDEALS AND FACTOR RINGS

**3.1. Pointed algebras and factor rings.** We want to define “factor rings” for a generalization of  $\mathbb{Z}$ -based rngs.

**Definition 3.1.** Let  $R$  be a finitely generated commutative  $\mathbb{Z}$ -algebra which is a free  $\mathbb{Z}$ -module of finite rank with basis  $B$ . We call  $(R, B)$  a *pointed  $\mathbb{Z}$ -algebra*.

We have *structure constants* as in the special case of  $\mathbb{Z}$ -based rngs.

**Definition 3.2.** Let  $(R, B)$  be a pointed  $\mathbb{Z}$ -algebra and  $I$  an ideal in  $R$ . If  $B' \subset B$  is a subset of the basis such that the factor ring  $R/I$  is a free  $\mathbb{Z}$ -module with basis  $B' + I$ , then we call  $(R/I, B' + I)$  a *factor ring* of  $(R, B)$ .

Let  $(R, B)$  be a  $\mathbb{Z}$ -based rng and  $I$  an ideal in  $R$ . If  $B' \subset B$  is a subset of the basis such that the factor ring  $(R/I, B' + I)$  with basis given by  $B'$  is a  $\mathbb{Z}$ -based rng, then we call  $(R/I, B' + I)$  a *factor ring* of  $(R, B)$ .

Remark that for most ideals, the factor module  $R/I$  is not free anymore. And even if it has no torsion, in most cases there will be no involution  $\sim$  on  $R/I$  (of course this depends on the choice of subset in  $B$ ).

Examples for such factor rings will be given in the following sections: There is an interesting ideal in the representation ring of the quantum double of a finite group which yields a pointed  $\mathbb{Z}$ -algebra with nonnegative structure constants. Also, each Hadamard matrix corresponds to a ring which is a factor ring of a ring with nonnegative structure constants.

**3.2. An ideal which yields a free factor ring.** When  $R$  is the representation ring of some Hopf algebra, then in many cases there is a one dimensional “sign” representation which acts via tensor product as a permutation on the irreducible representations. We can use this to become a new pointed  $\mathbb{Z}$ -algebra with nonnegative structure constants:

**Theorem 3.3.** *Let  $(R, B)$  be a  $\mathbb{Z}$ -based rng and  $d \in B$  an element of order 2 which acts via multiplication as a permutation on  $B$ . Consider the ideal*

$$I := \langle 1 - d \rangle \trianglelefteq R.$$

*Then there is a subset  $B' \subseteq B$  such that  $(R/I, B' + I)$  is a pointed  $\mathbb{Z}$ -algebra with nonnegative structure constants.*

*Proof.* By assumption, multiplication by  $d$  is a bijection on the basis. Since  $d^2 = 1$ , we get a partition of  $B$  into equivalence classes  $\{b, db\}$ , equivalent elements are equal in  $R/I$ . Let  $B'$  be a set of representatives. The  $\mathbb{Z}$ -module  $R' := R/I$  is free, and the set  $\{a + I \mid a \in B'\}$  is a basis: assume that  $R'$  has a torsion element  $\bar{w}$ ,  $w \in R$ , so  $n\bar{w} = 0$  for some  $n \in \mathbb{N}$ . Then  $nw \in I$ , i.e.  $nw$  may be written as  $r(1 - d)$  for an  $r = \sum_i a_i b_i \in R$ ,  $a_i \in \mathbb{Z}$ , if the  $b_i$  are the elements of  $B$ . Let  $\hat{i}$  be the index such that  $b_{\hat{i}} = b_i \cdot d$ . Then

$$nw = r(1 - d) = \sum_i a_i b_i (1 - d) = \sum_i (a_i - a_{\hat{i}}) b_i,$$

and hence each  $(a_i - a_{\hat{i}})$  is divisible by  $n$ . Let us write  $(a_i - a_{\hat{i}}) =: nc_i$  with  $c_i \in \mathbb{Z}$ . Moreover,

$$\sum_{b_i \in B} (a_i - a_{\hat{i}}) b_i = \sum_{b_i \in B'} (a_i - a_{\hat{i}}) (b_i - b_{\hat{i}}) = n \sum_{b_i \in B'} c_i (b_i - b_{\hat{i}}) \in nI.$$

But a  $\mathbb{Z}$ -based rng is a free  $\mathbb{Z}$ -module. Therefore  $nr_1 = nr_2$  implies  $r_1 = r_2$  for all  $r_1, r_2 \in R$ . Because of  $nw \in nI$ , we get  $w \in I$ , i.e.  $\bar{w} = 0$ .

The structure constants  $\tilde{N}_{*,*}^*$  of  $R/I$  for  $b_i, b_j, b_k \in B'$  are

$$\tilde{N}_{b_i, b_j}^{\overline{b_k}} = N_{b_i, b_j}^{b_k} + N_{b_i, b_j}^{b_{\hat{k}}}$$

and therefore they are nonnegative.  $\square$

**3.3.  $\mathbb{Z}$ -based rngs as factor rings.** An interesting task is to find for a given  $\mathbb{Z}$ -based rng  $R'$  a  $\mathbb{Z}$ -based rng  $R$  with nonnegative structure constants such that  $R'$  is a factor ring of  $R$ .

Let  $W$  be the set of one dimensional subspaces of  $\mathbb{C}^n$  spanned by products of columns of  $s$ ,

$$W := \left\{ \left\langle \prod_{\nu=1}^r s_{i_\nu} \right\rangle \mid 0 \leq i_1, \dots, i_r < n, r \in \mathbb{N} \right\}.$$

Denote by  $K$  the field extension of  $\mathbb{Q}$  given by the entries of  $s$ , by  $\mathcal{O}$  its ring of algebraic integers (see proposition 2.10) and by  $C$  the set of all roots of unity in  $\mathcal{O}$  and 0, so  $C = \{0, \zeta, \dots, \zeta^q\}$  for some  $q \in \mathbb{N}$  and primitive  $q$ -th root of unity  $\zeta$ .

**Lemma 3.4.** *If  $W$  is finite, then there are  $\mu_0, \dots, \mu_{n-1} \in \mathcal{O}$  such that  $s_{ij} \in \mu_j C$  for all  $0 \leq i, j < n$ .*

*Proof.* Choose  $i_1, i_2, j$  with  $s_{i_2, j} \neq 0$  and  $\mu := s_{i_1, j} s_{i_2, j}^{-1}$ . Since  $W$  is finite, the powers  $s_j^{k_1}, s_j^{k_2}$  of the  $j$ -th column are linearly dependent for some  $k_1, k_2 \in \mathbb{N}$ , so we have

$$s_{i_1, j}^{k_1} + \lambda s_{i_1, j}^{k_2} = 0, \quad s_{i_2, j}^{k_1} + \lambda s_{i_2, j}^{k_2} = 0$$

for  $0 \neq \lambda \in \mathbb{C}$ . Hence

$$\mu^{k_1} s_{i_2, j}^{k_1} + \lambda \mu^{k_2} s_{i_2, j}^{k_2} = 0, \quad \mu^{k_1} s_{i_2, j}^{k_1} + \lambda \mu^{k_1} s_{i_2, j}^{k_2} = 0$$

which implies  $(\mu^{k_1} - \mu^{k_2}) \lambda s_{i_2, j}^{k_2} = 0$ , i.e.  $\mu$  is a root of unity,  $\mu \in C$ .  $\square$

If for example all entries of  $\frac{1}{\mu} s$  are roots of unity for some  $\mu \in \mathbb{N}$ , then the columns of  $\frac{1}{\mu} s$  generate (multiplicatively) a finite group. In this case  $R$  is a factor ring of a subring of  $\mu$  times this group ring. More generally, for such a construction we need at least the finiteness of the set  $W$ .

**Theorem 3.5.** *Let  $(R, B)$  be a  $\mathbb{Z}$ -based rng with  $s$ -matrix  $s$ . Assume that the columns  $s_0, \dots, s_{n-1}$  are of the form  $s_i = \mu_i v_i$  with  $v_i \in C^n$  and  $0 \neq \mu_i \in \mathbb{Z}$ . Then  $R$  is a factor ring of a pointed  $\mathbb{Z}$ -algebra with nonnegative structure constants.*

*Proof.* For all  $a \in C$  we have  $-a \in C$ , so without loss of generality we may choose  $\mu_i > 0$  for all  $i$ . Let  $G$  be the multiplicative semigroup generated by the columns of  $s$ . Define a partial order  $\leq$  on  $C^n$  by

$$v \leq w \iff \mathbb{N}v \subseteq \mathbb{N}w$$

for  $v, w \in C^n$ . Let  $M$  be the set of maximal elements of  $G$  with respect to this order (they exist, because all entries of an element of  $G$  are of the form  $\lambda \xi$  with  $\lambda \in \mathbb{Z}$  and  $\xi$  a nonnegative power of  $\zeta$ ). The multiplicative semigroup  $H$  generated by  $v_0, \dots, v_{n-1} \in C^n$  is finite because  $C^n$  is finite. For each  $v \in H$ , there may be several elements  $w_i$  of  $M$  with  $w_i \leq v$ .

Let  $y_i$  be the number of elements of the multiplicative subsemigroup of  $H$  generated by  $v_i$ . So

$$H = \left\{ \prod_i^r v_i^{a_i} \mid 0 < a_i \leq y_i \text{ for all } i \right\}.$$

Let  $v$  be any element of  $G$ , we write it in the form  $v = \prod_i \mu_i^{a_i} v_i^{a_i}$ . Define  $c_i$  to be the number in  $0, \dots, y_i - 1$  with  $c_i \equiv a_i \pmod{y_i}$ .

Then  $v = \prod_i \mu_i^{a_i - c_i} t$  for  $t = \prod_i \mu_i^{c_i} v_i^{c_i}$ . But  $t \geq v$ , so for all elements of  $G$  we find a greater element in the finite subset

$$U = \left\{ \prod_i^r s_i^{a_i} \mid 0 < a_i \leq y_i \text{ for all } i \right\},$$

in particular,  $M \subseteq U$  and it is finite. We need a set of generators for our  $\mathbb{Z}$ -algebra. For this we take  $M$  and for all  $a_1 v, a_2 v \in M$ ,  $v \in H$ ,  $a_1, a_2 \in \mathbb{N}$  we add the element  $\gcd(a_1, a_2) v$ . We iterate this until no more new elements

appear and denote the resulting set by  $M''$ . (This iteration terminates because  $M$  is finite.)

It can happen that the product of two elements of  $M''$  is greater than all elements of  $M''$ , so we add all these products and do the above iteration for  $M''$  instead of  $M$  again. Since the set of elements of  $\mathbb{N}H$  that are greater or equal to all elements of  $M$  is finite, this process terminates. We need this construction because we have to show that we do not leave the  $\mathbb{Z}$ -lattice spanned by the columns of  $s$ :

Since the structure constants are integers, all elements of  $G$  are in this lattice; the only critical point is when we add  $\gcd(a_1, a_2)v$ . So as above, let  $a_1v, a_2v \in M$ ,  $v \in H$ ,  $a_1, a_2 \in \mathbb{N}$  and  $d := \gcd(a_1, a_2)$ . Suppose

$$a_1v = \sum_i q_i s_i, \quad a_2v = \sum_i r_i s_i,$$

for some  $q_i, r_i \in \mathbb{Z}$ . Consider  $dv$ , which is the new element we want to add to  $M$ . We have

$$dv = \sum_i \frac{q_i d}{a_1} s_i = \sum_i \frac{r_i d}{a_2} s_i$$

and since the  $s_i$  form a basis, we get  $\frac{q_i d}{a_1} = \frac{r_i d}{a_2}$  for all  $i$ . This means  $r_i \frac{a_1}{a_2} = q_i \in \mathbb{Z}$ , so  $\frac{a_2}{d}$  divides  $r_i$  and  $dv$  is in the lattice.

Let  $M'$  be the finite set of now uniquely determined maximal elements of  $M''$ . The product of two elements of  $M'$  is not greater than any element of  $M'$ . Remark that the columns of  $s$  are included in  $M'$ , because they are not smaller than any linear combination with integer coefficients of the  $s_i$  and every element of  $M'$  is such a linear combination. The  $s_i$  have already been in  $M$  at the beginning. Let  $G'$  be the multiplicative semigroup generated by  $M'$ .

Let  $R'$  be the free  $\mathbb{Z}$ -module with basis  $M'$  (view  $M'$  as a “formal” basis, so  $R'$  has rank  $|M'|$ ). We define the multiplication on  $R'$  in the following way. If  $v, w \in M'$ , then as elements of  $G'$  we have  $v \cdot w = u \in G'$  and there exists a unique  $x \in M'$  with  $x \geq u$ , so  $u = \mu x$  for some  $\mu \in \mathbb{N}$ . We set  $v \cdot w := \mu x$  in  $R'$ . This multiplication is associative, because the multiplication of  $G'$  is associative. Also, the structure constants of  $R'$  are nonnegative because  $\mu \in \mathbb{N}$ .

To get  $R$  as a factor rng of  $R'$ , we only need to see that the elements of  $M'$  are in the  $\mathbb{Z}$ -lattice spanned by the columns of  $s$ , which we have proved above. The ideal  $I$  in  $R'$  is generated by  $x - \sum_i q_i s_i$  if  $\sum_i q_i s_i$  is the linear decomposition of  $x$  in  $R$  with respect to the columns of  $s$ . We take the set  $B'$  of columns of  $s$  as a subset of  $M'$ . Then  $(R, B) \cong (R'/I, B' + I)$ .  $\square$

*Remark 3.6.* The ring  $R'$  of the last theorem is a subring of the semigroupring  $\mathbb{Z}[H]$  of  $H$ .

Now we can conclude:

**Proposition 3.7.** *Let  $(R, B)$  be a  $\mathbb{Z}$ -based rng with  $s$ -matrix  $s$ ,  $\zeta$  a complex  $q$ -th root of unity and  $C' := \{\zeta, \dots, \zeta^q\}$ . Assume that the columns*

$s_0, \dots, s_{n-1}$  are of the form  $s_i = \mu_i v_i$  with  $v_i \in C^m$  and  $0 \neq \mu_i \in \mathbb{Z}$ . Then  $R$  is a factor ring of a subring with nonnegative structure constants of  $k$  times a group ring for  $k = |\mu_i|$  for all  $i$ .

*Proof.* We use the notation of the proof of Theorem 3.5. Since  $0 \notin C'$ , the set  $H$  is a group. By proposition 2.7 we have  $|\mu_i| = |\mu_j|$  for all  $i, j$  and since without loss of generality  $\mu_i > 0$  for all  $i$ , they are all equal to some  $k \in \mathbb{N}$ . So all elements of  $M'$  lie in  $k\mathbb{Z}[H]$ , in particular  $R' \leq k\mathbb{Z}[H]$ .  $\square$

**Question 3.8.** Assume that the above set of spaces  $W$  is finite and that  $K = \mathbb{Q}[\zeta]$  for some root of unity  $\zeta$ . Is it true that  $R$  is a subalgebra of a factor ring of a pointed  $\mathbb{Z}$ -algebra with nonnegative structure constants?

To prove this, it would be enough to construct a matrix  $s'$  starting from  $s$  satisfying the assumptions of Theorem 3.5 by using the fact that  $\mathcal{O} = \mathbb{Z}[\zeta]$  is a pointed  $\mathbb{Z}$ -algebra with basis given by some powers of  $\zeta$ .

#### 4. EXAMPLES: SOME MODULAR TENSOR CATEGORIES

**4.1. Quantum doubles of finite groups.** Let  $G$  be a finite group and  $k$  an algebraically closed field of characteristic 0. The quantum double  $D(G)$  of  $G$  is a Hopf algebra isomorphic to  $F(G) \otimes k[G]$  as a vector space, where  $F(G)$  is the algebra of functions on  $G$  into  $k$  and  $k[G]$  is the group algebra. Its algebra and coalgebra structure will not be used here, see [2], 3.2 for more details or [6] for a twisted version.

The category of finite dimensional representations  $\mathcal{R}ep_f(D(G))$  of  $D(G)$  is a modular tensor category (see [2]). Its Grothendieck ring  $R(D(G))$  is a fusion algebra, so it is a  $\mathbb{Z}$ -based ring with nonnegative structure constants. There is an explicit formula for the corresponding  $s$ -matrix.

The irreducible representations of  $D(G)$  are parametrized by pairs  $(\bar{g}, \chi)$  where  $\bar{g}$  is a conjugacy class of  $G$  and  $\chi$  an irreducible character of the centralizer of  $g$  in  $G$ . We choose one representative  $g$  for each class  $\bar{g}$  and denote the irreducible modules by  $(g, \chi)$ . So we may identify the basis elements of  $R(D(G))$  with pairs  $(g, \chi)$ .

Let  $e$  denote the identity element of  $G$  and  $R := R(D(G))$ . Assume the existence of an irreducible  $D(G)$ -module  $(e, \chi)$  with  $(e, \chi) \cdot (e, \chi) = (e, 1)$ , where “ $\cdot$ ” is the tensor product in  $R$ . This just means that  $\chi^2 = 1$ , so for example if  $G = S_n$ , we can choose the sign character for  $\chi$ . Let

$$I := \langle (e, 1) - (e, \chi) \rangle \trianglelefteq R,$$

so in  $R/I$  we identify each basis element  $(g, \psi)$  with  $(e, \chi)(g, \psi)$ .

Assume that  $(e, \chi)$  acts via multiplication as a permutation on the basis elements of  $R(D(G))$ . Then by theorem 3.3, the factor ring  $R(D(G))/I$  is a pointed  $\mathbb{Z}$ -algebra with nonnegative structure constants.

*Example 9.* Let  $G = S_3$ . The  $s$ -matrix of  $R(D(G))/I$  is

$$\begin{pmatrix} 1 & 2 & 3 & 2 & 2 & 2 \\ 1 & 2 & -3 & 2 & 2 & 2 \\ 1 & 2 & 0 & -1 & -1 & -1 \\ 1 & -1 & 0 & -1 & -1 & 2 \\ 1 & -1 & 0 & -1 & 2 & -1 \\ 1 & -1 & 0 & 2 & -1 & -1 \end{pmatrix}.$$

**4.2. Affine Kac-Moody algebras.** let  $\mathfrak{g}(A)$  be an affine Kac-Moody algebra of arbitrary untwisted type  $X_l^{(1)}$  or  $A_{2l}^{(2)}$  and denote the fundamental weights by  $\Lambda_0, \dots, \Lambda_l$ .

For each positive integer  $k$ , let  $P_+^k$  be the finite set

$$P_+^k := \left\{ \sum_{j=0}^l \lambda_j \Lambda_j \mid \lambda_j \in \mathbb{Z}, \lambda_j \geq 0, \sum_{j=0}^l a_j^\vee \lambda_j = k \right\}.$$

Kac and Peterson defined a natural  $\mathbb{C}$ -representation of the group  $\mathrm{SL}_2(\mathbb{Z})$  on the subspace spanned by the affine characters of  $\mathfrak{g}$  which are indexed by  $P_+^k$ . The image of  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  under this representation is determined in Theorem 13.8 of [11]. This is the so-called *Kac-Peterson matrix*:

$$S_{\Lambda, \Lambda'} = c \sum_{w \in W^\circ} \det(w) \exp \left( - \frac{2\pi i (\bar{\Lambda} + \bar{\rho} \mid w(\bar{\Lambda}' + \bar{\rho}))}{k + h^\vee} \right),$$

where  $\Lambda, \Lambda'$  runs through  $P_+^k$ ,  $(\cdot \mid \cdot)$  is the normalized bilinear form of chapter 6 of [11],  $W^\circ$  is the Weyl group of  $\mathfrak{g}^\circ$  and  $c \in \mathbb{C}$  some constant which is unimportant for us, since we want to use the matrix in Verlinde's formula (section 2.2, (1)).

Each of these matrices defines a  $\mathbb{Z}$ -based rng (even a fusion algebra). A classification of the matrices belonging to type  $X_l^{(1)}$  up to isomorphism was given by Gannon in [9].

*Example 10.* Lusztig [13] gives an interpretation of the Fourier matrix for dihedral groups via the above fusion algebra of type  $A_1^{(1)}$ , level  $k$ . Denote this algebra by  $R$ . Then the Fourier matrix is the  $S$ -matrix of a factor ring of the algebra belonging to a closed subset of  $R \otimes R$ .

*Example 11.* Geck and Malle [10] use a similar construction (they choose another ideal) to become the Fourier matrix for the dihedral group with non-trivial automorphism. The resulting algebra has negative structure constants, so it is a  $\mathbb{Z}$ -based ring.

*Example 12.* The author [7] has interpreted many of the Fourier matrices associated to the exceptional spetses as factor rings of algebras corresponding to Kac-Peterson matrices.

For example, let  $R$  be the affine ring corresponding to the Kac-Peterson matrix of type  $B_3^{(1)}$  and level 2. The algebras corresponding to the Fourier matrix of some families of the exceptional complex reflection group  $G_{24}$  are isomorphic to  $R'/I$  for some ideal  $I$ , where  $R'$  is a subring (to a closed subset) of  $R \otimes Z_4$  ( $Z_4$  is the group ring of the cyclic group with 4 elements).

## 5. EXAMPLE: HADAMARD MATRICES

**5.1. General case.** Let  $s$  be a Hadamard matrix of order  $4k$ , i.e.

$$s \in \{\pm 1\}^{4k \times 4k}, \quad ss^T = 4kI, \quad k \in \mathbb{N},$$

where  $I$  denotes the identity matrix. We assume  $k > 1$  and  $s_{i,1} = 1$  for all  $i$ .

**Proposition 5.1.** *The  $\mathbb{Z}$ -lattice  $R$  spanned by the columns of  $k \cdot s$  is closed under componentwise multiplication, i.e., if  $b_1, \dots, b_{4k}$  are the columns of  $ks$ , then*

$$b_i b_j = \sum_m N_{ij}^m b_m$$

for some  $N_{ij}^m \in \mathbb{Z}$ . We have

$$N_{ij}^m = \frac{1}{4} \sum_l s_{li} s_{lj} s_{lm}$$

for all  $i, j, m$ .

*Proof.* Choose  $i, j, m$  such that  $|\{1, i, j, m\}| = 4$ . Then without loss of generality (after permuting some rows), the transposed of the columns  $i, j, m$  will be

$$\begin{array}{cccccccc} - & \dots & - & - & \dots & - & - & \dots & - & + & \dots & + & + & \dots & + & + & \dots & + \\ - & \dots & - & - & \dots & - & + & \dots & + & - & \dots & - & - & \dots & - & - & \dots & - \\ - & \dots & - & + & \dots & + & - & \dots & - & - & \dots & - & + & \dots & + & - & \dots & - \\ \underbrace{\hspace{1.5cm}}_a & \underbrace{\hspace{1.5cm}}_{k-a} & \underbrace{\hspace{1.5cm}}_{k-a} & \underbrace{\hspace{1.5cm}}_a & \underbrace{\hspace{1.5cm}}_{k-a} & \underbrace{\hspace{1.5cm}}_a & \underbrace{\hspace{1.5cm}}_a & \underbrace{\hspace{1.5cm}}_a & \underbrace{\hspace{1.5cm}}_{k-a} \end{array}$$

for some  $1 \leq a < k$ , where “+”, “−” stands for “1” respectively “−1”. That  $a$  and  $k - a$  appear several times comes from the mutual orthogonality of the three columns: Column  $i$  and  $j$  must have exactly  $k$  common −1’s because they are orthogonal. Suppose that column  $m$  has  $a$  −1’s at these positions. Then it has to have  $k - a$  −1’s where column  $i$  is −1 and column  $j$  is 1 because it is orthogonal to column  $i$ , and so on.

Counting together, the product of the three columns is a vector with  $4a$  −1’s. This yields  $\sum_l s_{li} s_{lj} s_{lm} \equiv 4k - 8a \equiv 0 \pmod{4}$ . If one of the indices  $i, j, m$  is 1 or if two of them are equal, then  $\sum_l s_{li} s_{lj} s_{lm}$  reduces to orthogonality or to counting −1’s in a column.

So the numbers  $N_{ij}^m$  defined above are integers. That these numbers are the structure constants of the lattice spanned by the columns of  $s$  follows from  $s^{-1} = \frac{1}{4k} s^T$  and Verlinde’s formula (1).  $\square$

*Remark 5.2.* Theorem [17], 9.9 states that  $p_{ijlm} \equiv 4k \pmod{8}$ , where

$$p_{ijlm} = \left| \sum_q s_{qi} s_{qj} s_{ql} s_{qm} \right|.$$

In particular for  $m = 1$  we get the absolute values of 4 times the structure constants. So the last proposition is a corollary of this Theorem. The *profile* of a Hadamard matrix is the map  $\pi : \mathbb{N} \rightarrow \mathbb{Z}$ , where  $\pi(q)$  is the number of sets  $\{i, j, l, m\}$  of four distinct columns such that  $p_{ijlm} = q$ . Profiles are used to test if Hadamard matrices are not equivalent and they also give some results about the number of equivalence classes.

**Corollary 5.3.** *A Hadamard matrix gives rise to a  $\mathbb{Z}$ -based rng  $R$ .*

*Proof.* By the previous proposition, we may associate a  $\mathbb{Z}$ -algebra to each Hadamard matrix. It has a basis consisting of elements  $b_i$  with  $b_i^2 = kb_1$ . Also,  $N_{i,j}^1 = 0$  for  $i \neq j$ . So it is a  $\mathbb{Z}$ -based rng with trivial involution  $\sim$  and is of the same form as example 3.  $\square$

**Definition 5.4.** We call the  $\mathbb{Z}$ -algebra  $R$  of proposition 5.1 the  $\mathbb{Z}$ -based rng or the *ring* associated to the Hadamard matrix  $s$ .

**Lemma 5.5.** *If there are  $i, j, l$  such that  $b_i b_j = kb_l$  and  $i, j, l \neq 1$  then  $k$  is even and  $k \cdot \mathbb{Z}[\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}]$  is a subalgebra of  $R$ .*

*Proof.* By the assumptions,  $\{b_1, b_i, b_j, b_l\}$  is a closed subset. Up to the factor  $k$ , this set forms a group isomorphic to  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ . When  $k$  is odd, such a closed subset can not exist, see corollary 5.10.  $\square$

To describe the structure constants  $N_{ij}^l$  of the ring of a Hadamard matrix in another way, we introduce the following notation. Let

$$\xi_i := \{j \mid s_{ji} = -1\}$$

and  $\Delta$  denote the symmetric difference of sets.

**Lemma 5.6.** *For  $i, j$  with  $|\{1, i, j\}| = 3$  we have*

$$|\xi_i| = |\xi_i \Delta \xi_j| = 2|\xi_i \cap \xi_j| = 2k.$$

*Proof.* Since  $s_{i1} = 1$  for all  $i$ , we have  $|\xi_i| = |\xi_j| = 2k$  for  $i, j \neq 1$ . If in addition  $i \neq j$  then  $|\xi_i \Delta \xi_j| = 2k$  since  $ss^T$  is diagonal. Hence  $|\xi_i \cap \xi_j| = k$ .  $\square$

**Lemma 5.7.** *For  $i, j, l$  with  $|\{1, i, j, l\}| = 4$  we have*

$$|\xi_i \Delta \xi_j \Delta \xi_l| = 4|\xi_i \cap \xi_j \cap \xi_l| \quad \text{and} \quad |\xi_i \cap \xi_j \cap \xi_l| = |\xi_i \setminus (\xi_j \cup \xi_l)|.$$

*Proof.* Let  $a := |\xi_i \cap \xi_j \cap \xi_l|$ . The proof of proposition 5.1 gives the first equality. Further,

$$|\xi_i| = k + |\xi_i \cap \xi_j| = |\xi_i \setminus (\xi_j \cup \xi_l)| + |\xi_i \cap \xi_j| + |\xi_i \cap \xi_l| - a,$$

so  $2k = |\xi_i \setminus (\xi_j \cup \xi_l)| + k + k - a$ .  $\square$

Thus for the structure constants we get (this is also a consequence of the proof of Proposition 5.1):

**Lemma 5.8.** *For  $i, j, m$  with  $|\{1, i, j, m\}| = 4$  we have*

$$N_{ij}^m = k - \frac{1}{2}|\xi_i \Delta \xi_j \Delta \xi_m| = k - 2|\xi_i \cap \xi_j \cap \xi_m|.$$

**5.2. Odd  $k$ .** The most important open question concerning Hadamard matrices is certainly the existence of such matrices in all dimensions  $4k$ . If  $k$  is even, then we can take a matrix of size  $2k \times 2k$  and tensor it with a  $2 \times 2$ -matrix. So from now on,  $k$  will always be odd.

From Lemma 5.8, we then get:

**Corollary 5.9.** *Let  $k$  be odd. If  $i, j, m$  are such that  $|\{1, i, j, m\}| = 4$ , then*

$$N_{ij}^m \equiv 1 \pmod{2}.$$

*In fact, we know even more:  $N_{ij}^m \in \{-k+2, -k+4, \dots, k-2\}$ .*

Corollary 5.9 gives us all the information needed to determine the closed subsets of  $R$ :

**Corollary 5.10.** *If  $k$  is odd, then the closed subsets of  $R$  are exactly the sets  $\{b_1, b_i\}$  for  $1 \leq i \leq 4k$  and the set of all basis elements.*

*Proof.* Suppose  $B'$  is a closed subset with two different elements  $b_i$  and  $b_j$ , both unequal to  $b_1$ . Then by corollary 5.9,  $N_{ij}^m \neq 0$  for all  $m$  with  $|\{1, i, j, m\}| = 4$ . But  $B'$  is closed, so  $b_m \in B'$  for all  $m \neq 1$ . Further,  $b_i^2 = kb_1$  which implies  $b_1 \in B'$ .  $\square$

Since the ring of a Hadamard matrix is a  $\mathbb{Z}$ -based rng, Proposition 2.4 applies:

**Proposition 5.11.** *Let  $(R, B)$  be a  $\mathbb{Z}$ -based rng of rank  $4k$ . Assume that  $\sim$  is the identity on  $B$ , that  $k \cdot 1 = b_1 \in B$  and that*

$$N_{ii}^j = \delta_{1,j}k$$

*for all  $i, j$ . Then there exists a Hadamard matrix of size  $4k \times 4k$ .*

*Proof.* By proposition 2.4, the  $s$ -matrix  $s$  of  $R$  has orthogonal rows. Multiplication of elements of  $B$  corresponds to componentwise multiplication of columns of  $s$ . Hence  $N_{ii}^j = \delta_{1,j}k$  means  $s_{li}^2 = s_{li}k = k^2$  for all  $i, l$ . So the entries of  $s$  have to be equal to  $\pm k$ .  $\square$

The last proposition is a good motivation to examine the properties of the structure constants. By Corollary 5.9 we already know what the structure constants are modulo 2. If it is possible to lift them to the 2-adic numbers  $\mathbb{Z}_2$ , then the existence of a Hadamard matrix also follows. Of course, the same holds for the entries of a Hadamard matrix: we know them modulo 2 and cannot lift them to  $\mathbb{Z}_2$ .

The matrix  $N_i := (N_{ij}^m)_{j,m}$  describes the operation of the  $i$ -th column on the other columns by componentwise multiplication ( $N_i$  is the image of  $b_i$

in the regular representation of  $R$ ). Since we have  $b_i^2 = kb_1$ , the square of this matrix is a scalar matrix:  $N_i^2 = k^2 I$ . So we get:

**Lemma 5.12.**  $\sum_m (N_{ij}^m)^2 = k^2$  for all  $i, j$ .

And using this, it is easy to prove (where the  $r$ -th triangular number is defined to be  $\frac{r(r+1)}{2}$ ):

**Proposition 5.13.** *For given  $i, j$  with  $|\{1, i, j\}| = 3$ , the number of different possible multisets for the absolute values of the structure constants  $|N_{ij}^m|$ ,  $m \notin \{1, i, j\}$  is less or equal to the number of partitions of the  $\frac{k-3}{2}$ -th triangular number into nonzero triangular numbers.*

*Proof.* We have  $4k - 3$  indices  $m$  such that the structure constants  $N_{ij}^m$  are unequal to 0 and in this case they are odd. We are considering multisets, so the number we are interested in is the number of elements of

$$\begin{aligned} & \{(i_1, \dots, i_{4k-3}) \mid 1 \leq i_1 \leq \dots \leq i_{4k-3}, \quad 2 \nmid i_m \quad \forall m, \quad \sum_{m=1}^{4k-3} i_m^2 = k^2\} \\ &= \{\bar{i} \in \mathbb{Z}^{4k-3} \mid 0 \leq i_1 \leq \dots \leq i_{4k-3}, \quad \sum_{m=1}^{4k-3} (2i_m + 1)^2 = k^2\} \\ &= \{\bar{i} \in \mathbb{Z}^{4k-3} \mid 0 \leq i_1 \leq \dots \leq i_{4k-3}, \quad \sum_{m=1}^{4k-3} i_m^2 + i_m = \lambda^2 + \lambda\} \end{aligned}$$

for  $\lambda := \frac{k-3}{2}$ . □

Proposition 5.11 may be understood as follows. The associativity of  $R$  is equivalent to the fact that the matrices  $N_i$  commute. Since they are semisimple, with eigenvalues  $\pm k \in \mathbb{Z}$ , they are simultaneously diagonalizable. Commutativity of  $R$  corresponds to the symmetry of the  $N_i$ .

Conversely, for the same reason, if a Hadamard matrix is given, then we may recover it from the structure constants of its ring. And if  $k \equiv 1 \pmod{3}$ , then we may even reduce them modulo 3. Indeed, over  $\mathbb{F}_3$  the matrices still commute, the eigenvalues are  $\pm 1$  and  $N_i^2 = I$  for all  $i$ . So they are still simultaneously diagonalizable which yields the Hadamard matrix over  $\mathbb{F}_3$  without loss of information.

**Proposition 5.14.** *If  $k \equiv 1 \pmod{3}$  then it is possible to compute the Hadamard matrix from the structure constants modulo 3 of its ring.*

Taking the structure constants of a Hadamard matrix modulo 2 yields an  $\mathbb{F}_2$ -algebra (this is trivial, but it should be stated for the sake of completeness).

**Corollary 5.15.** *The vector space  $\mathbb{F}_2^{4k}$  with basis  $v_1, \dots, v_{4k}$  and multiplication defined by  $v_i v_j = \sum_m N_{ij}^m v_m$  where*

$$N_{ij}^1 = N_{1i}^j = N_{i1}^j = \delta_{ij}, \quad N_{ij}^m = 1$$

for  $|\{1, i, j, m\}| = 4$ , is an associative commutative algebra.

We return to Hadamard matrices over  $\mathbb{Z}$ . The submatrix  $\hat{N}_i$  of  $N_i$  corresponding to the indices not equal to  $1, i$  ( $i \neq 1$ ) is a square matrix with entries in  $\mathbb{Z}$  which has zeroes only on its diagonal. Moreover, it has orthogonal rows,

$$\hat{N}_i \hat{N}_i^T = k^2 I.$$

This is a generalization of weighing matrices or more precisely of conference matrices (see [17], p. 145).

**Proposition 5.16.** *Let*

$$W_i := \begin{pmatrix} \hat{N}_i + I & \hat{N}_i - I \\ \hat{N}_i - I & -\hat{N}_i - I \end{pmatrix},$$

*$i \neq 1$ . Then  $W_i \cdot W_i^T = (2k^2 + 2)I$ . (Where ‘ $I$ ’ always denotes the appropriate identity matrix.)*

So  $W_i$  is an orthogonal  $(8k - 4) \times (8k - 4)$ -matrix with entries in  $\mathbb{Z} \setminus \{0\}$ .

*Example 13.* Let  $k = 3$ , so  $s$  is a  $12 \times 12$  Hadamard matrix. The matrices  $W_i$  of proposition 5.16 are then  $20 \times 20$  Hadamard matrices. Note that this is the only case in which the  $W_i$  have entries in  $\{\pm 1\}$ .

**5.3. Hadamard matrices as factor rings.** By proposition 3.7, the ring of a Hadamard matrix is a factor ring of a ring with nonnegative structure constants. We want to look at this ring more explicitly.

Denote the cyclic group with two elements by  $Z_2$ . The ring  $R$  of a Hadamard matrix of size  $4k \times 4k$  is a factor ring of a subring of  $k \cdot R'$ , where  $R'$  is the group algebra  $\mathbb{Z}[Z_2^{4k-2}]$ . We can see this in the following way.

Let  $v$  be the matrix defined by  $v_{ij} := \frac{1-s_{ij}}{2} \in \mathbb{F}_2$  and denote the columns of  $v$  by  $v_1, \dots, v_n$ . Remember that we have  $s_{i,1} = 1$  for all  $i$ , so  $v_1 = 0$ .

**Proposition 5.17.** *The rank of  $v$  is less or equal to  $4k - 2$ .*

*Proof.* The first column is 0, so it suffices to give a linear combination of the other columns to prove the claim. But the number of  $-1$ ’s in each row of  $s$  is even, so  $\sum_{i=2}^n v_i = 0$ .  $\square$

So the vector space spanned by the columns of  $v$  has dimension at most  $4k - 2$ , as an abelian group it is isomorphic to a subgroup of  $Z_2^{4k-2}$ . Componentwise multiplication of column vectors of  $s$  corresponds to addition of column vectors of  $v$ .

Let  $G$  be the multiplicative semigroup generated by the columns of  $ks$ . Then  $\mathbb{Z}[G]$  is a subalgebra of  $\mathbb{Z}[Z_2^{4k-2}]$ .

By identifying each vector of  $G$  with its linear combination with respect to the columns of  $ks$  (this is a  $\mathbb{Z}$ -linear combination, because the structure constants are integers), we get an ideal  $I$  in  $\mathbb{Z}[G]$  such that  $R \cong \mathbb{Z}[G]/I$ .

This proves:

**Proposition 5.18.** *The ring  $(R, B)$  of a Hadamard matrix  $s$  of size  $4k \times 4k$  is a factor ring of the algebra  $\mathbb{Z}[H]$*

$$R \cong \mathbb{Z}[H]/I,$$

where  $H$  is the multiplicative semigroup spanned by the columns of  $s$ .  $\mathbb{Z}[H]$  is a subalgebra of the group algebra  $\mathbb{Z}[Z_2^{4k-2}]$ . The ideal  $I$  is given by

$$I := \langle w - \sum_i \mu_{w,i} b_i \mid w \in G \rangle$$

where  $G$  is the multiplicative semigroup spanned by the columns of  $ks$ , and the  $\mu_{w,i}$  are given by the linear decomposition of  $w$  with respect to the columns of  $ks$ .

An interesting question is: What are the ideals of different Hadamard matrices (of the same size)? If we do the same construction for  $\hat{R} := \mathbb{Z}[Z_2^{4k}]$ , then in  $\hat{R}_{\mathbb{C}} = \hat{R} \otimes_{\mathbb{Z}} \mathbb{C}$ , we always get the same ideal. The proof looks complicated, but it is completely straightforward.

**Proposition 5.19.** *In the algebra  $\hat{R}_{\mathbb{C}} = \mathbb{C}[Z_2^{4k}]$ , the above ideals are equal for all Hadamard matrices.*

*Proof.* The ideals of  $\hat{R}_{\mathbb{C}}$  are easy to describe. For this, it suffices to give a basis of primitive idempotents;  $\hat{R}_{\mathbb{C}}$  is isomorphic to  $\mathbb{C}^{2^n}$ ,  $n = 4k$  with componentwise multiplication. Then it is obvious that the ideals are in bijection with the subsets of  $\{1, \dots, 2^n\}$ .

We identify  $Z_2$  with  $\mathbb{Z}^{\times}$  and view the columns  $s_1, \dots, s_n$  of  $s$  as elements of  $Z_2^n$ , so there exists a map  $\tau : \{1, \dots, n\} \rightarrow \{1, \dots, 2^n\}$  such that  $s_i = b_{\tau(i)}$  if  $b_1, \dots, b_{2^n}$  are the elements of  $Z_2^n$ . Define

$$\sigma : Z_2^n \rightarrow \mathbb{F}_2^n, \quad (\varepsilon_1, \dots, \varepsilon_n) \mapsto (\delta_{\varepsilon_1, -1}, \dots, \delta_{\varepsilon_n, -1}).$$

The primitive idempotents are given by the character table  $T$  of  $Z_2^n$  which is the  $n$ -th tensor power of the character table of  $Z_2$ . More explicitly,

$$T_{i,j} = (-1)^{(\sigma(b_i) | \sigma(b_j))}$$

where  $(\cdot | \cdot)$  is the standard inner product on  $\mathbb{F}_2^n$ . So the primitive idempotents are

$$c_i := \frac{1}{2^n} \sum_j T_{i,j} b_j$$

for  $i = 1, \dots, 2^n$  and we also get  $b_i = \sum_j T_{i,j} c_j$ . Now in  $\mathbb{C}^{\times n}$ , we have decompositions

$$b_i = \sum_{j=1}^n \lambda_{ij} s_j$$

for some suitable  $\lambda_{i,j}$ . Denote  $\lambda_i := (\lambda_{i1}, \dots, \lambda_{in})$ . Then

$$\lambda_i = s^{-1} b_i = \frac{1}{n} s^T b_i$$

because  $s^{-1} = \frac{1}{n}s^T$ . The ideal  $I$  is generated by all elements  $b_i - \sum_{j=1}^n \lambda_{ij}s_j$  and with respect to the basis  $c_1, \dots, c_{2^n}$ , this becomes

$$I = \langle \sum_m (T_{i,m} - \sum_{j=1}^n \lambda_{ij}T_{\tau(j),m})c_m \mid i = 1, \dots, 2^n \rangle.$$

Now it suffices to check that the coefficient  $T_{i,m} - \sum_{j=1}^n \lambda_{ij}T_{\tau(j),m}$  is zero exactly for  $m$  such that there exists  $\mu$  with  $\sigma(b_m)_\nu = \delta_{\mu,\nu}$  for all  $\nu$ . But for such an  $m$ ,

$$T_{i,m} = (-1)^{\sigma(b_i)_\mu} = b_{i,\mu}$$

and also  $T_{\tau(j),m} = b_{\tau(j),\mu}$ . The assertion now follows from the orthogonality of  $s$ .  $\square$

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